

On stochastic heat equation with measure initial data

Jingyu Huang

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Abstract

The aim of this short note is to obtain the existence, uniqueness and moment upper bounds of the solution to a stochastic heat equation with measure initial data, without using the iteration method in [1, 2, 3].

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1 Introduction

Consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u + b(u) + \sigma(u)\dot{W} \quad (1)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d (d \geq 1)$ where \mathcal{L} is the generator of a Lévy process $X = \{X_t\}_{t \geq 0}$. \dot{W} is a centered Gaussian noise with covariance formally given by

$$\mathbb{E}(\dot{W}(t, x)\dot{W}(s, y)) = \delta(s - t)f(x - y),$$

where f is some nonnegative and nonnegative definite function whose Fourier transform is denoted by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\xi} dx$$

in distributional sense, and δ denotes the Dirac delta function at 0. For some technical reasons, we will assume that f is lower semicontinuous (see Lemma 4 below).

Let Φ be the Lévy exponent of X_t , we will assume that

$$\exp(-\operatorname{Re}\Phi) \in L^t(\mathbb{R}^d) \text{ for all } t > 0. \quad (2)$$

Thus according to Proposition 2.1 in [5], X_t has a transition function $p_t(x)$ and we can (and will) find a version of $p_t(x)$ which is continuous on $(0, \infty) \times \mathbb{R}^d$ and uniformly continuous for all $(t, x) \in [\eta, \infty) \times \mathbb{R}^d$ for every $\eta > 0$, and that p_t vanishes at infinity for all $t > 0$.

The initial condition $u(0, \cdot)$ is assumed to be a (positive) measure $\mu(\cdot)$ such that

$$\int_{\mathbb{R}^d} p_t(x-y)\mu(dy) < \infty \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (3)$$

To avoid trivialities, we assume that $\mu(\cdot) \not\equiv 0$.

Using iteration method, the existence, uniqueness and some moment bounds of the solution have been obtained in [1, 2, 3] for the case $b \equiv 0$ and for some specific choice of \mathcal{L} . However, these approaches rely on the structure (or asymptotic structure) of $p_t(x)$. In this article, we will study the equation (1) with also a Lipschitz drift term b and establish the existence, uniqueness and p -th moment upper bound, without using the iteration method in [1, 2, 3], also, our criteria only need some integrability of the Lévy exponent.

To state the result, let us recall that by a solution u to (1) we mean a mild solution. That is, (i) u is a predictable random field on a complete probability space $\{\Omega, \mathcal{F}, P\}$, with respect to the Brownian filtration generated by the cylindrical Brownian motion defined by $B_t(\phi) := \int_{[0,t] \times \mathbb{R}^d} \phi(y)W(ds, dy)$, for all $t \geq 0$ and measurable $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(y)\phi(z)f(y-z)dydz < \infty$; and (ii) for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$, the following equation holds a.s.

$$\begin{aligned} u(t, x) = & \int_{\mathbb{R}^d} p_t(x-y)\mu(dy) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b(u(s, y))dyds \\ & + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s, y))W(ds, dy). \end{aligned} \quad (4)$$

where $p_t(x)$ is the transition function for X_t and the stochastic integral above is in the sense of Walsh [6]. The following theorem is the main result of this paper.

Theorem 1. *Assume that the initial condition satisfies (3) and assume that*

$$\Upsilon(\beta) := \sup_{t>0} \int_0^t \int_{\mathbb{R}^d} \exp \left[-2s\text{Re}\Phi \left(\left(1 - \frac{s}{t}\right)\xi \right) - 2(t-s)\text{Re}\Phi \left(\frac{s}{t}\xi \right) \right] e^{-2\beta(t-s)} \hat{f}(\xi) d\xi ds < \infty \quad (5)$$

and

$$\tilde{\Upsilon}(\beta) := \int_{\mathbb{R}^d} \frac{\hat{f}(\xi) d\xi}{\beta + \text{Re}\Phi(\xi)} < \infty \quad (6)$$

for any $\beta > 0$. And assume that σ and b are Lipschitz functions with Lipschitz coefficients $L_\sigma, L_b > 0$ respectively. Then there exists a unique mild solution to equation (1). Moreover, define

$$\bar{\gamma}(p) := \limsup_{t \rightarrow \infty} \frac{1}{t} \sup_{x \in \mathbb{R}^d} \log \left\| \frac{u(t, x)}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)}, \quad (7)$$

where

$$\tau = \max \left\{ \frac{|b(0)|}{L_b}, \frac{|\sigma(0)|}{L_\sigma} \right\}. \quad (8)$$

Then,

$$\bar{\gamma}(p) \leq \inf \{ \beta > 0 : B(\beta, p) < 1 \} \quad \text{for all integers } p \geq 2, \quad (9)$$

where

$$B(\beta, p) := \frac{L_b}{\beta} + \frac{z_p L_\sigma}{(2\pi)^{d/2}} \left(\sqrt{\frac{\tilde{\Upsilon}(\beta)}{2}} + \sqrt{\Upsilon(\beta)} \right), \quad (10)$$

and z_p denotes the largest positive zero of the Hermite polynomial He_p .

Remark 2. If we choose \mathcal{L} to be the generator of an a -stable Lévy process D_θ^a for $1 < a < 2$, where θ is the skewness and $|\theta| < 2-a$ (see [2]), or the Laplacian $\frac{1}{2}\Delta$ ($a = 2$), then the classical Dalang's condition

$$\int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{1 + |\xi|^a} < \infty \quad (11)$$

implies condition (5), since in this case $\text{Re}\Phi(\xi) = C|\xi|^a$ for some $C > 0$. Also, in the case $d = 1$ and \dot{W} is a space-time white noise, that is, $f(\xi) \equiv 1$, condition (6) clearly guarantees that (2) holds.

Remark 3. (Borrowed from [5, Remark 1.5]). Recall that

$$He_k(x) = 2^{-k/2} H_k(x/\sqrt{2}) \quad \text{for all integers } k \geq 0 \text{ and } x \in \mathbb{R},$$

where $\{H_k\}_{k=0}^\infty$ is defined uniquely via the following:

$$e^{-2xt-t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \quad (t > 0, x \in \mathbb{R}).$$

2 Proof of Theorem 1

In the proof of Theorem 1 we will need two results about taking Fourier transforms, which we now state.

Lemma 4 (Corollary 3.4 in [5]). *Assume that f is lower semicontinuous, then for all Borel probability measures ν on \mathbb{R}^d ,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \nu(dx) \nu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) |\hat{\nu}(\xi)|^2 d\xi.$$

Lemma 5. *If f is lower semicontinuous, then*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y_1) p_s * \mu(y_1) p_{t-s}(x-y_2) p_s * \mu(y_2) f(y_1-y_2) dy_1 dy_2 \\ & \leq \frac{[p_t * \mu(x)]^2}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left[-2s \text{Re}\Phi \left(\left(1 - \frac{s}{t}\right) \xi \right) - 2(t-s) \text{Re}\Phi \left(\frac{s}{t} \xi \right) \right] \hat{f}(\xi) d\xi. \end{aligned}$$

Proof. We begin by noting that

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y_1) p_s * \mu(y_1) p_{t-s}(x-y_2) p_s * \mu(y_2) f(y_1-y_2) dy_1 dy_2 \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1) p_s(y_1-z_1)}{p_t(x-z_1)} \frac{p_{t-s}(x-y_2) p_s(y_2-z_2)}{p_t(x-z_2)} f(y_1-y_2) dy_1 dy_2 \\ & \quad \times p_t(x-z_1) p_t(x-z_2) \mu(dz_1) \mu(dz_2), \end{aligned}$$

and as a function of y , the quotient $\frac{p_{t-s}(x-y)p_s(y-z)}{p_t(x-z)}$ is the probability density of the Lévy bridge $\tilde{X}_{z,x,t} = \{\tilde{X}_{z,x,t}(s)\}_{0 \leq s \leq t}$ which is at z when $s = 0$ and at x when $s = t$. Actually, $\tilde{X}_{z,x,t}(s)$ can be written as

$$\begin{aligned}\tilde{X}_{z,x,t}(s) &= X_s - \frac{s}{t}X_t + z + \frac{s}{t}(x - z) \\ &= (1 - \frac{s}{t})X_s - \frac{s}{t}(X_t - X_s) + z + \frac{s}{t}(x - z),\end{aligned}$$

hence by the independence of increment of Lévy process, we have

$$\mathbb{E}e^{i\xi\tilde{X}_{z,x,t}(s)} = \exp\left(-s\Phi\left((1 - \frac{s}{t})\xi\right) - (t-s)\Phi\left(-\frac{s}{t}\xi\right)\right)e^{i(z + \frac{s}{t}(x-z))}.$$

Thus, an application of Lemma 4 to $\nu_j(dy) = \frac{p_{t-s}(x-y)p_s(y-z_j)}{p_t(x-z_j)}dy$, $j = 1, 2$, yields

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1)p_s(y_1-z_1)}{p_t(x-z_1)} \frac{p_{t-s}(x-y_2)p_s(y_2-z_2)}{p_t(x-z_2)} f(y_1-y_2) dy_1 dy_2 \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}e^{i\xi\tilde{X}_{z_1,x,t}(s)} \overline{\mathbb{E}e^{i\xi\tilde{X}_{z_2,x,t}(s)}} \hat{f}(\xi) d\xi \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left[-2s\operatorname{Re}\Phi\left((1 - \frac{s}{t})\xi\right) - 2(t-s)\operatorname{Re}\Phi\left(-\frac{s}{t}\xi\right)\right] \hat{f}(\xi) d\xi,\end{aligned}$$

which proves the lemma. \square

To prove Theorem 1, we first define a norm for all $\beta, p > 0$ and all predictable random fields $v := v(t, x)$,

$$\|v\|_{\beta,p} = \sup_{t \geq 0} e^{-\beta t} \sup_{x \in \mathbb{R}^d} \|v(t, x)\|_{L^p(\Omega)}. \quad (12)$$

Let $\mathcal{B}_{\beta,p}$ denote the collection of all predictable random fields $v := \{v(t, x)\}_{t \geq 0, x \in \mathbb{R}^d}$ such that $\|v\|_{\beta,p} < \infty$. We note that after the usual identification of a process with its modifications, $\mathcal{B}_{\beta,p}$ is a Banach space (see Section 5 in [5]).

Proof of Theorem 1. We use Picard iteration. Set

$$\begin{aligned}u^0(t, x) &:= p_t * \mu(x), \\ u^{n+1}(t, x) &:= p_t * \mu(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b(u^n(s, y))dyds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u^n(s, y))W(ds, dy).\end{aligned}$$

We first show that whenever β is chosen such that $B(\beta, p) < 1$, where $B(\beta, p)$ is defined in (10), then, for any $n \geq 1$,

$$\left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta,p} < \infty. \quad (13)$$

Note that by the dominated convergence theorem, the condition $B(\beta, p) < 1$ can be achieved if β is sufficiently large.

Recall that τ is defined in (8). We start with the inequality

$$\begin{aligned} \frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t * \mu(x)} &\leq 1 + \left| \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} \frac{b(u^n(s, y))}{\tau + p_s * \mu(y)} dy ds \right| \\ &\quad + \left| \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)(\tau + p_s * \mu(y))}{\tau + p_t * \mu(x)} \frac{\sigma(u^n(s, y))}{\tau + p_s * \mu(y)} W(ds, dy) \right|. \end{aligned}$$

(13) is clearly true for $n = 0$. Using induction, assume (13) is true for some n , using Burkholder inequality (see [4]) and the assumption on σ and b , we obtain

$$\begin{aligned} &\left\| \frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)} \\ &\leq 1 + L_b \int_0^t \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} \left\| \frac{\tau + |u^n(s, y)|}{\tau + p_s * \mu(y)} \right\|_{L^p(\Omega)} dy ds \\ &\quad + z_p L_\sigma \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1)(\tau + p_s * \mu(y_1))}{\tau + p_t * \mu(x)} \frac{p_{t-s}(x-y_1)(\tau + p_s * \mu(y_1))}{\tau + p_t * \mu(x)} \right. \\ &\quad \times \left. \left\| \frac{\tau + |u^n(s, y_1)|}{\tau + p_s * \mu(y_1)} \right\|_{L^p(\Omega)} \left\| \frac{\tau + |u^n(s, y_2)|}{\tau + p_s * \mu(y_2)} \right\|_{L^p(\Omega)} f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2}, \end{aligned}$$

multiplying both sides by $e^{-\beta t}$ and applying Minkowski's inequality to the third summand above we obtain

$$\begin{aligned} &e^{-\beta t} \left\| \frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)} \\ &\leq 1 + L_b \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p} \int_0^t \int_{\mathbb{R}^d} e^{-\beta(t-s)} \frac{p_{t-s}(x-y)[\tau + p_s * \mu(y)]}{\tau + p_t * \mu(x)} dy ds \\ &\quad + z_p L_\sigma \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p} \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\beta(t-s)} p_{t-s}(x-y_1) p_{t-s}(x-y_2) f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2} \\ &\quad + z_p L_\sigma \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p} \\ &\quad \times \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\beta(t-s)} \frac{p_{t-s}(x-y_1) p_s * \mu(y_1)}{p_t * \mu(x)} \frac{p_{t-s}(x-y_2) p_s * \mu(y_2)}{p_t * \mu(x)} f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2} \\ &:= 1 + I_1 + I_2 + I_3, \end{aligned}$$

where in obtaining I_2 and I_3 above, we have used the bound

$$\frac{p_{t-s}(x-y)\tau}{\tau + p_t * \mu(x)} \leq p_{t-s}(x-y) \quad \text{and} \quad \frac{p_{t-s}(x-y)p_s * \mu(y)}{\tau + p_t * \mu(x)} \leq \frac{p_{t-s}(x-y)p_s * \mu(y)}{p_t * \mu(x)}. \quad (14)$$

We will estimate I_1, I_2, I_3 separately. For I_1 , the semigroup property of $p_t(x)$ yields

$$I_1 \leq \frac{L_b}{\beta} \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p}.$$

For I_2 , an application of Lemma 4 to $\nu(dy) = p_{t-s}(x-y)dy$ yields

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\beta(t-s)} p_{t-s}(x-y_1) p_{t-s}(x-y_2) f(y_1-y_2) dy_1 dy_2 ds \\ &= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} e^{-2(t-s)\operatorname{Re}\Phi(\xi)} \hat{f}(\xi) d\xi e^{-2\beta(t-s)} ds \leq \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\xi) d\xi}{\beta + \operatorname{Re}\Phi(\xi)}, \end{aligned}$$

thus we obtain

$$I_2 \leq z_p L_\sigma \left(\frac{1}{2(2\pi)^d} \tilde{\Upsilon}(\beta) \right)^{1/2} \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p}.$$

Finally, an application of Lemma 5 yields

$$I_3 \leq z_p L_\sigma \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p} \left(\frac{1}{(2\pi)^d} \Upsilon(\beta) \right)^{1/2}.$$

Combining the estimates for I_1, I_2, I_3 , we arrive at

$$\left\| \frac{\tau + |u^{n+1}|}{\tau + p * \mu} \right\|_{\beta, p} \leq 1 + B(\beta, p) \left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p},$$

where $B(\beta, p)$ is defined in (10). Using the iteration, we see that (13) holds for all $n \geq 1$ if $B(\beta, p) < 1$.

The same technique applied to $\frac{u^{n+1}(t, x) - u^n(t, x)}{\tau + p_t * \mu(x)}$ yields that

$$\left\| \frac{u^{n+1} - u^n}{\tau + p * \mu} \right\|_{\beta, p} \leq B(\beta, p) \left\| \frac{u^n - u^{n-1}}{\tau + p * \mu} \right\|_{\beta, p},$$

and if β is chosen such that $B(\beta, p) < 1$, we will obtain that

$$\sum_{n=1}^{\infty} \left\| \frac{u^n - u^{n-1}}{\tau + p * \mu} \right\|_{\beta, p} < \infty.$$

Therefore, we can find a predictable random field $u^\infty \in \mathcal{B}_{\beta, p}$ such that $\lim_{n \rightarrow \infty} u^n = u^\infty$ in $\mathcal{B}_{\beta, p}$. It is easy to see that this u^∞ is a solution to equation (4), and uniqueness is checked by a standard argument.

To prove (9), we note that since $u \in \mathcal{B}_{\beta, p}$ for those β such that $B(\beta, p) < 1$,

$$\sup_{x \in \mathbb{R}^d} \left\| \frac{u(t, x)}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)} \leq \sup_{x \in \mathbb{R}^d} \frac{\tau}{\tau + p_t * \mu(x)} + C e^{\beta t}$$

for some $C > 0$ which does not depend on t , thus (9) is proved and the proof of Theorem 1 is complete. \square

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Jingyu Huang

Department of Mathematics

University of Utah

Salt Lake City, UT 84112-0090

Email: jhuang@math.utah.edu

URL: <http://www.math.utah.edu/~jhuang/>